Conformal Field Theory and Gravity

Solutions to Problem Set 13 Fall 2024

1. Hawking-Page in Three Dimensions

(a) The Euclidean BTZ black hole with the identification $t_E \sim t_E + \beta$ has the metric, stated in the exercise set,

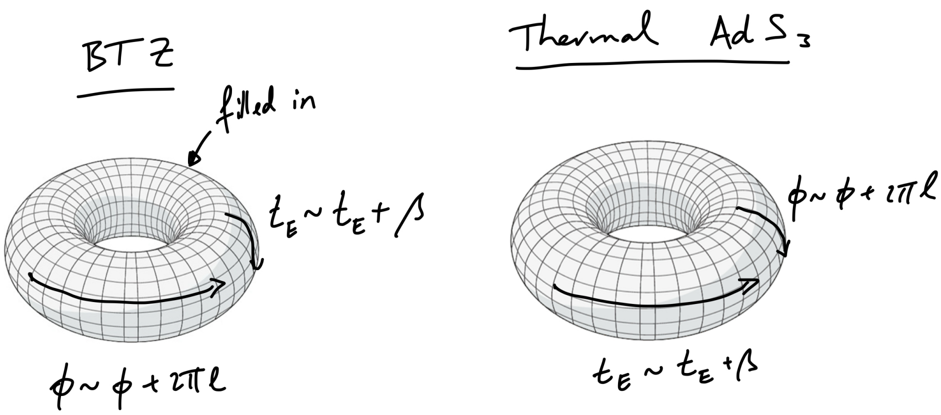
$$ds^2 = \ell^2 \left[\left(r^2 - 8M\right) dt_E^2 + \frac{dr^2}{r^2 - 8M} + r^2 d\phi^2 \right]. \tag{1}$$

Because of the identifications $\phi \sim \phi + 2\pi\ell$ and $t_E \sim t_E + \beta$, this can be understood, for a given and fixed r, as a torus. When $r \to \sqrt{8M}$, the thermal circle shrinks to 0 whereas the ϕ circle remains finite in size. Thus, we interpret this manifold as a filled in torus, where it is the thermal circle which is filled in.

The thermal AdS_3 with identification $t_E \sim t_E + \beta$ and $\phi \sim \phi + 2\pi\ell$ has the metric

$$ds^2 = \left(1 + \frac{r^2}{\ell^2}\right) dt_E^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\phi^2 \tag{2}$$

Again, we interpret it as a torus, but this time as $r \to 0$, it is the ϕ circle which shrinks while the thermal circle remains finite in size. Both manifolds are depicted in the following figure:



To compute the on-shell action of the thermal AdS manifold, we first rescale to make t_E have the same periodicity as BTZ, so that we can use its on-shell action. This means rescaling $(t_E, \phi) \to \frac{2\pi\ell}{\beta}(t_E, \phi)$, so that the filled in circle has periodicity $2\pi\ell \cdot \frac{2\pi\ell}{\beta}$, obtaining

$$S_{(th)}^{\text{on-shell}} = S_{(bh)}^{\text{on-shell}}\big|_{\beta \to \frac{4\pi^2\ell^2}{\beta}} = -\frac{\beta c}{12\ell} \tag{3}$$

(b) We are interested in the limit where $G_N \to 0$ ($\ell_p \to \infty$, $M_P \to 0$), meaning $c \to \infty$. In this limit, the partition function reduces to $Z = e^{-S_{\text{min}}^{\text{on-shell}}}$ where

$$S_{\text{min}}^{\text{on-shell}} = \min(S_{(th)}^{\text{on-shell}}, S_{(bh)}^{\text{on-shell}}) \tag{4}$$

At small temperature $T \ll 1$ ($\beta \gg 1$), the minimum is given by thermal AdS, whereas at high temperatures, it is given by BTZ. The transition takes place at

$$S_{(th)}^{\text{on-shell}} = S_{(bh)}^{\text{on-shell}} \implies T = T^* = \frac{1}{2\pi\ell} \tag{5}$$

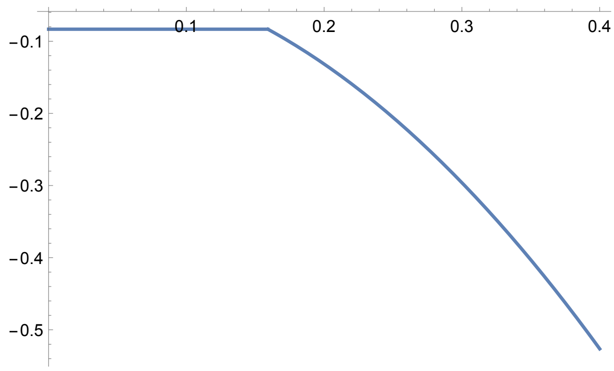
We can thus write compactly

$$S_{\text{min}}^{\text{on-shell}} = -\frac{c}{12\ell T}\Theta(T^* - T) - \frac{\pi^2 \ell c T}{3}\Theta(T - T^*) \tag{6}$$

where Θ is the Heaviside function. Thus,

$$F = -T \log Z = T S_{\text{min}}^{\text{on-shell}} = -\frac{c}{12\ell}\Theta(T^* - T) - \frac{\pi^2 \ell c T^2}{3}\Theta(T - T^*) \tag{7}$$

which can be plotted as (here I'm plotting F/c with $\ell = 1$):



Note that since $\partial F/\partial T$ is discontinuous, the transition at $T = T^*$ is a first order phase transition.

(c) The entropy is calculated from the partition function thanks to

$$S = \frac{\partial}{\partial T}(T \log Z) = -\frac{\partial}{\partial T}(T S_{\text{min}}^{\text{on-shell}}) = \frac{2\pi^2 \ell c}{3} T \; \Theta(T - T^*) \tag{8}$$

(note that the entropy vanishes for thermal AdS, as expected).

(d) To compute the mean energy in the canonical ensemble, use

$$E(\beta) = -\frac{\partial}{\partial \beta}\log Z = \frac{\partial}{\partial \beta} S_{\text{min}} = -\frac{c}{12\ell}\Theta(T^* - T) + \frac{\pi^2 \ell c T^2}{3}\Theta(T - T^*) \tag{9}$$

The first term corresponds to the Casimir energy in the CFT (in this sense, thermal AdS corresponds to the CFT vacuum), whereas the second term gives the spacetime energy $M(T)$, using the relation between M and T derived in Problem set 11.

(e) Inverting this relation, we obtain (for large T and large E where $T > T^*$)

$$1/T = \beta = \sqrt{\frac{\pi^2 c\ell}{3E}} \tag{10}$$

and thus,

$$S(E) = \frac{2\pi^2 \ell c}{3}\beta = 2\pi\sqrt{\frac{cE\ell}{3}} \tag{11}$$

(f) This gives precisely the Cardy formula, which predicts the exponentially growing density of states $\rho(E) = e^{S(E)}$ that we obtained in a previous exercise set.

2. Small and large black holes in general d

(a) We may determine the temperature of the black hole by relating it to the horizon radius r_h, which is given by the larger root of $f(r_h) = 0$, i.e.,

$$1 - \frac{\mu}{r_h^{d-2}} + \frac{r_h^2}{L^2} = 0. \tag{12}$$

Since the metric near the horizon at $r \approx r_h$ reads

$$ds^2 = f'(r_h)(r - r_h)d\tau^2 + \frac{1}{f'(r_h)(r - r_h)}dr^2 + r_h^2 d\Omega_{d-1}^2, \tag{13}$$

the temperature of the black hole is given by (recall problem set 4)

$$T = \frac{|f'(r_h)|}{4\pi} = \frac{dr_h^2 + (d-2)L^2}{4\pi L^2 r_h}. \tag{14}$$

On the other hand, the associated temperature of thermal AdS is given by the inverse of the compactified Euclidean time direction. This difference originates from the different topology of the solutions: for black holes, the thermal circle is contractible, while for thermal AdS it is not.

(b) Solving for $r_h(T)$, we find two solutions

$$r_h^{\pm} = \frac{2\pi L^2 T}{d}\left(1 \pm \sqrt{1 - \frac{d(d-2)}{4\pi^2 L^2 T^2}}\right) \tag{15}$$

and we get $T = T_{min}$ when the discriminant is 0. We see that there is no black hole solution when $T < T_{min}$. Nonetheless, thermal AdS remains a solution (it satisfies Einstein's equations with a negative cosmological constant for every T).

(c) The two solutions have been previously found

$$r_h^{\pm} = \frac{2\pi L^2 T}{d}\left(1 \pm \sqrt{1 - \frac{d(d-2)}{4\pi^2 L^2 T^2}}\right) \tag{16}$$

(d) Consider the bare gravitational action in $D = d+1$ dimensions in Euclieann signature

$$S_{bulk} + S_{GHY} = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^D x \sqrt{g}\,(R - 2\Lambda) + \frac{1}{\kappa^2}\int_{\partial\mathcal{M}} d^d x \sqrt{\gamma} K \tag{17}$$

where $\mathcal{M}$ is the spacetime bulk and $\partial\mathcal{M}$ is the asymptotic AdS boundary, and we introduced $\Lambda = -\frac{(D-1)(D-2)}{2L^2}$. Note that since we will only consider differences in actions, we need not renormalise the expression above.
A solution of Einstein's equation satisfies $R = \frac{2D}{D-2}\Lambda$, and thus we can substitute this to find

$$S_{bulk} = \frac{1}{2\kappa^2}\int d^D x \sqrt{g}\,(R - 2\Lambda) = \frac{1}{2\kappa^2}\frac{4\Lambda}{D-2}\int d^D x \sqrt{g} \tag{18}$$

Therefore

$$\begin{aligned}\Delta S_{bulk} &= -\frac{(D-1)}{\kappa^2 L^2}\left(\mathrm{Vol}_{BH} - \mathrm{Vol}_{AdS}\right) \\ &= -\frac{(D-1)}{\kappa^2 L^2}\beta \mathrm{Vol}(S^{d-1}) \lim_{r\to\infty}\left(\int_{r_h}^{r} dr' r'^{d-1} - \int_0^r dr' r'^{d-1}\right) \\ &= \frac{(D-1)}{\kappa^2 L^2}\beta \mathrm{Vol}(S^{d-1})\frac{r_h^d}{d} = \frac{r_h^d}{\kappa^2 L^2}\beta \mathrm{Vol}(S^{d-1})\end{aligned} \tag{19}$$

Now let us consider the relevant quantities to compute the boundary action. The normal vector to the boundary at a distance r is $n = f^{1/2}\partial_r$, the metric determinant is $\sqrt{\gamma} = f^{1/2} r^{d-1}$, and the extrinsic curvature density is given by $K = \frac{1}{2}\gamma^{\mu\nu}\mathcal{L}_n \gamma_{\mu\nu} = \frac{1}{2} f^{1/2}\partial_r f + \ldots$ where the dots indicate terms that vanish when we do the angular integrals.

At large radii, $r \gg L$, we have

$$\sqrt{\gamma}_{BH} = \sqrt{\gamma}_{AdS} - \frac{\mu L}{2} \tag{20}$$

$$K_{BH} = K_{AdS} + (d-1)\frac{\mu L}{2 r_h^d} \tag{21}$$

where $\sqrt{\gamma}_{AdS} \approx \frac{r^d}{L}$ and $K_{AdS} \approx 1/L$. Now we can determine the difference in boundary actions

$$\begin{aligned}\Delta S_{GHY} &= \frac{\beta \mathrm{Vol}(S^{d-1})}{\kappa^2}\lim_{r\to\infty}\left(\sqrt{\gamma}_{AdS}(d-1)\frac{\mu L}{2 r_h^d} - K_{AdS}\frac{\mu L}{2}\right) \\ &= \frac{(d-2)\beta \mathrm{Vol}(S^{d-1})}{2\kappa^2}\mu = \frac{(d-2)\beta \mathrm{Vol}(S^{d-1})}{2\kappa^2 L^2}\left(r_h^{d-2}L^2 + r_h^d\right)\end{aligned} \tag{22}$$

where we found μ from (12). hence the action is proportional to the volume of spacetime. This is infinite, due to the infinite range of r, but the difference between the volumes of two asymptotically AdS geometries is a finite quantity, which we can compute as

$$\Delta F = \frac{r_h^{d-2}}{2\kappa^2}\mathrm{Vol}(S^{d-1})\left(1 - \frac{r_h^2}{L^2}\right). \tag{23}$$

(e) For $r_h < L$, the difference ΔF of the free energies is positive, and therefore thermal AdS is preferred. For $r_h > L$, the Schwarzschild black hole is preferred. For $r_h = L$, there is a phase transition, the Hawking–Page phase transition, with transition temperature

$$T_{HP} = \frac{1}{2\pi L}(d-1).$$

Small black holes have $r_h < L$, hence their free energy is always greater than thermal AdS, and such configurations never dominate the partition function.

(f) The specific heat can be related to the free energy by

$$C = T\frac{d^2 F}{dT^2} \tag{24}$$

To conclude, we see that large black holes are stable, while small ones decay into thermal AdS for $T \leq \frac{d-1}{2\pi L}$ and into large black holes for $T > \frac{d-1}{2\pi L}$.

(g) As we take $L \to \infty$, we recover the flat space limit, where $T_{HP} = 0$, hence no phase transition occurs, and the partition function is always dominated by the black hole solution, of which there is now a single one, with free energy $F = \frac{r_h^{d-2}}{2\kappa^2}\mathrm{Vol}(S^{d-1})$. This is always positive and therefore black holes will tend to decay to the empty flat space solution, with $F = 0$.